Chapter 10

Absolute-Value Equations and Inequalities

In This Chapter

- Changing from an absolute value equation to separate linear equations
- Recognizing when no solution is possible
- Transforming an absolute value inequality into one or two statements

The *absolute value* function actually measures a distance. How far is the number from 0? So the direction of a value — right or left of zero — doesn't make any difference in the world of absolute value. The symbol that signifies that you're performing the absolute-value function is two vertical lines — you sandwich the number to be operated upon between the lines. Absolute value strips away negative signs. Because of this, when solving equations or inequalities involving absolute value, you have to account for the original number having been either positive or negative.

Acting on Absolute-Value Equations

Before tackling the inequalities, take a look at absolute-value equations. An equation such as $|x| = 7$ is fairly easy to decipher. It's asking for values of x that give you a 7 when you put it in the absolute-value symbol. Two answers, 7 and –7, have an absolute value of 7. Those are the only two answers. But what about something a bit more involved, such as $|3x + 2| = 4$? The equation is true if the sum of $3x$ and 2 is equal to +4. But it's also true if the sum of $3x$ and 2 is equal to –4. The two possibilities for the sum result in two possibilities for the value of x.

To solve an absolute-value equation of the form $|ax + b| = c$, change the absolute-value equation to two equivalent linear equations and solve them.

$|ax + b| = c$ is equivalent to $ax + b = c$ or $ax + b = -c$. Notice that the left side is the same in each equation. The c is positive in the first equation and negative in the second because the expression inside the absolute-value symbol can be positive or negative — absolute value makes them both positives when it's performed.

Solve for x in $|3x + 2| = 4$.

1. **Rewrite as two linear equations.**

   $$3x + 2 = 4 \text{ or } 3x + 2 = -4$$

2. **Solve for the value of the variable in each of the equations.**

   Subtract 2 from each side in each equation: $3x = 2$ or $3x = -6$.

   Divide each side in each equation by 3: $x = \frac{2}{3}$ or $x = -2$.

3. **Check.**

   If $x = -2$, then $|3(-2) + 2| = |-6 + 2| = |-4| = 4$.
   If $x = \frac{2}{3}$, then $\left|3\left(\frac{2}{3}\right) + 2\right| = |2 + 2| = 4$.
   They both work.

In the next example, you see the equation set equal to 0. For these problems, though, you don't want a number added to or subtracted from the absolute value on the same side of the equal sign. In order to use the rule for changing to linear equations, you have to have the absolute value by itself on one side of the equation.

Solve for x in $|5x - 2| + 3 = 0$.

1. **Get the absolute-value expression by itself on one side of the equation.**

Adding –3 to each side:

$$|5x-2|=-3$$

2. **Rewrite as two linear equations.**

   $5x-2=-3$ or $5x-2=+3$

3. **Solve the two equations for the value of the variable.**

   Add 2 to each side of the equations:

   $5x=-1$ or $5x=5$

   Divide each side by 5:

   $$x=-\frac{1}{5} \text{ or } x=1$$

4. **Check.**

   If $x=-\frac{1}{5}$ then, $\left|5\left(-\frac{1}{5}\right)-2\right|+3=|-1-2|+3=|-3|+3=6.$

   Oops! That's supposed to be a 0. Try the other one.

   If $x=1$, then $|5(1)-2|+3=|3|+3=6.$

   No, that didn't work either.

Now's the time to realize that the equation was impossible to begin with. (Of course, noticing this before you started would've saved time.) The definition of absolute value tells you that it results in everything being positive. Starting with an absolute value equal to –3 gave you an impossible situation to solve. No wonder you didn't get an answer!

Working Absolute-Value Inequalities

Absolute-value inequalities are just what they say they are — inequalities that have absolute-value symbols somewhere in the problem.

$|a|$ is equal to a if a is a positive number or 0. $|a|$ is equal to the opposite of a, or $-a$, if a is a negative number. So $|3|=3$ and $|-7|=-(-7)=7$.

Absolute-value equations and inequalities can look like the following:

$$|x+3|=5 \qquad |2x+3|>7 \qquad |5x+1|\le 9$$

Solving absolute-value inequalities brings two different procedures together into one topic. The first procedure involves the methods similar to those used to deal with absolute-value equations, and the second involves the rules used to solve inequalities. You might say it's the best of both worlds. Or you might not.

To solve an absolute-value inequality of the form $|ax+b|>c$, change the absolute-value inequality to two inequalities equivalent to that original problem and solve them: $|ax+b|>c$ is equivalent to $ax + b > c$ or $ax + b < -c$. Notice that the inequality symbol is reversed with the $-c$.

Solve for x in $|2x-5|>7$.

1. **Rewrite as two inequalities.**

   $2x - 5 > 7$ or $2x - 5 < -7$

2. **Solve each inequality.**

   Add 5 to each side in each inequality:

   $2x > 12$ or $2x < -2$

   Divide through by 2:

   $x > 6$ or $x < -1$

   In interval notation, that's $(-\infty, -1) \cup (6, \infty)$. (See Chapter 9 for more on interval notation.)

The answer seems to go in two different directions — and it does. You need numbers that get larger and larger to keep the result bigger than 7, and you need numbers that get smaller and smaller so that the absolute value of the small negative numbers is also bigger than 7. That's why, when doing the solving, you use both greater than the $+c$ and less than the $-c$ in the problem.

Now, consider the absolute-value inequality that is kept small. The result of performing the absolute value can't be too large — it has to be smaller than c.

To solve an absolute-value inequality of the form $|ax + b| < c$, change the absolute-value inequality to an equivalent compound inequality and solve it: $|ax + b| < c$ is equivalent to $-c < ax + b < c$.

Solve for x in $|5x + 1| \le 9$.

1. **Rewrite as two inequalities.**

   $$-9 \le 5x + 1 \le 9$$

2. **Solve the inequality.**

   Subtract 1 from each section:

   $$-10 \le 5x \le 8$$

   Now divide through by 5:

   $-2 \le x \le \frac{8}{5}$ or, in interval notation, $\left[-2, \frac{8}{5}\right]$

Notice that this problem had a less-than-or-equal-to symbol. The rules for *less than* or *greater than* are the same as those for the problems including the endpoints of the interval — when the numbers establishing the starting or ending points are included in the answer.